Section 17.2

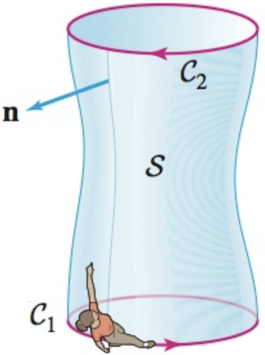
Stokes' Theorem

1 The Stokes' Theorem

Surface Orientations and Boundary Orientations

Suppose $\mathcal{S}$ is an oriented surface with unit normal vector $\vec{n}$.

Recall that the **boundary** $\partial\mathcal{S}$ consists of a set of closed curves.



We can orient $\partial\mathcal{S}$ as follows: *If you walk along $\partial\mathcal{S}$ with your head in the direction of $\vec{n}$, then $\mathcal{S}$ should be on your left.*

This is sometimes called a **right-hand-rule orientation**. If your right thumb points up (toward $\vec{n}$), then you should be able to curl your fingers from the direction of travel along $\partial\mathcal{S}$ inward toward $\mathcal{S}$.

Stokes' Theorem

Let $\mathcal{S}$ be an oriented surface with smooth, simple closed boundary curves. Let $\vec{\mathsf{F}}$ be a vector field whose components have continuous partial derivatives. Then

$$\oint_{\partial \mathcal{S}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} = \iint_{\mathcal{S}} \operatorname{curl}(\vec{\mathsf{F}}) \cdot d\vec{\mathsf{S}}$$

where the components of $\partial \mathcal{S}$ are oriented using a right-hand-rule orientation.

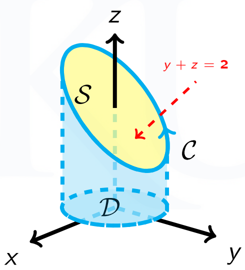
Green's Theorem is a special case of Stokes' Theorem. If $\mathcal{D}$ is a region in the plane and $\partial \mathcal{D}$ is given a right-hand-rule orientation (with $\vec{\mathsf{n}} = \vec{\mathsf{k}}$), then Stokes' Theorem becomes

$$\oint_{\partial \mathcal{D}} \vec{\mathsf{F}} \cdot d\vec{\mathsf{r}} = \iint_{\mathcal{D}} \operatorname{curl}(\vec{\mathsf{F}}) \cdot \vec{\mathsf{k}}\, dA$$

which is exactly Green's Theorem.

Stokes' Theorem: Examples

Example 1: Verify Stokes' Theorem for the vector field $\vec{F}(x, y, z) = \langle -y^2, x, z \rangle$ and the surface $\mathcal{S}$ obtained by intersecting the plane $y + z = 2$ and the solid cylinder $x^2 + y^2 \leq 1$.



Solution: First, $\text{curl}(\vec{F}) = \langle 0, 0, 1 + 2y \rangle$.

The surface $\mathcal{S}$ can be parametrized over the unit disk $\mathcal{D}$ by $G(x, y) = (x, y, 2 - y)$. The normal is

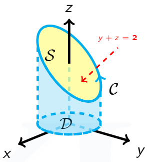
$$G_x \times G_y = \langle 1, 0, 0 \rangle \times \langle 0, 1, -1 \rangle = \langle 0, 1, 1 \rangle$$

which points upwards.

The double-integral side of Stokes' Theorem is

$$\underbrace{\iint_{\mathcal{S}} \text{curl}(\vec{F}) \cdot d\vec{S}}_{\text{Surface Integral}} = \iint_{\mathcal{D}} 1 + 2y \, dA = \int_0^{2\pi} \int_0^1 (1 + 2r\sin(\theta))\, r \, dr \, d\theta = \boxed{\pi.}$$

Example 1 (continued): For reference, $\vec{F}(x,y,z) = \langle -y^2, x, z \rangle$.



The boundary $\partial\mathcal{S}$ is the ellipse with parametrization

$$\vec{r}(t) = \langle \cos(t),\ \sin(t),\ 2-\sin(t) \rangle$$
$$\vec{r}\,'(t) = \langle -\sin(t),\ \cos(t),\ -\cos(t) \rangle$$

so $$\vec{F}(\vec{r}(t)) = \left\langle -\sin^2(t),\ \cos(t),\ 2-\sin(t) \right\rangle .$$

Therefore, the contour-integral side of Stokes' Theorem is

$$\overbrace{\oint_{\partial\mathcal{S}} \vec{F}\cdot d\vec{r}}^{\text{Line Integral}} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^{2\pi} \sin^3(t) + \cos^2(t) - 2\cos(t) + \sin(t)\cos(t)\, dt$$

$$= \int_0^{2\pi} \cos^2(t)\, dt = \boxed{\pi.}$$

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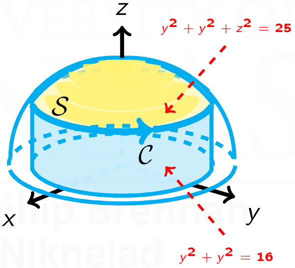
Stokes' Theorem: Examples

Example 2: Compute $\iint_{\mathcal{S}} \text{curl}(\vec{F}) \cdot d\vec{S}$ where $\vec{F}(x, y, z) = \langle xz, yz, xy \rangle$ and $\mathcal{S}$ is the part of the sphere $x^2 + y^2 + z^2 = 25$ inside the cylinder $x^2 + y^2 = 16$ above the xy-plane.

Solution: $\mathcal{C} = \partial\mathcal{S}$ can be parametrized as

$$\vec{r}(t) = \langle 4\cos(t),\ 4\sin(t),\ 3\rangle$$
$$\vec{r}\,'(t) = \langle -4\sin(t),\ 4\cos(t),\ 0\rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 12\cos(t),\ 12\sin(t),\ 16\cos(t)\sin(t)\rangle$$



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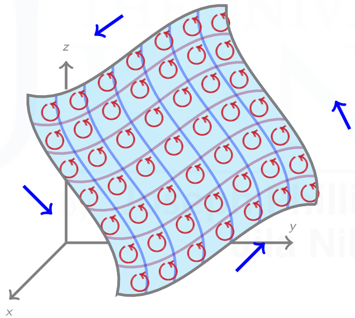
Using Stokes' Theorem,

$$\iint_{\mathcal{S}} \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^{2\pi} 0\, dt = 0.$$

Varying the radius of the cylinder would not change the answer of 0.

Curl and Circulation in Green and Stokes

Green's and Stokes' Theorems both say that if a vector field pushes stuff (counter)clockwise around the boundary of a surface $\mathbb{R}^2$, then it rotates stuff (counter)clockwise in the surface itself.



The circulation per unit area is $\text{curl}(\vec{F})_z$ (Green) or $\text{curl}(\vec{F}) \cdot \vec{n}$ (Stokes).

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2 Vector Potentials and Surface-Independence

Vector Potentials and Surface-Independence

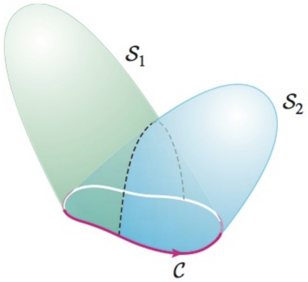
A **vector potential** for a vector field $\vec{F}$ is a vector field $\vec{A}$ such that

$$\vec{F} = \text{curl}(\vec{A}).$$

If $\vec{F}$ has a vector potential, then its integral over a surface $\mathcal{S}$ depends only on $\partial\mathcal{S}$, because Stokes' Theorem says that

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iint_{\mathcal{S}} (\nabla \times \vec{A}) \cdot d\vec{S} = \oint_{\partial\mathcal{S}} \vec{A} \cdot d\vec{r}.$$

For example, if $\vec{F} = \nabla \times \vec{A}$ then $\vec{F}$ has the same flux through the two surfaces $\mathcal{S}_1, \mathcal{S}_2$ shown to the right, because $\partial\mathcal{S}_1 = \partial\mathcal{S}_2$.



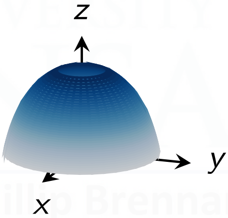
This is very useful: we can often compute flux through a complicated surface by replacing it with a simpler surface with the same boundary.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same boundaries (including orientation(s)), then

$$\iint_{\mathcal{S}_1} \nabla \times \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}} = \iint_{\mathcal{S}_2} \nabla \times \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}}.$$

Example 3: Consider the surface $\mathcal{S}$ in $\mathbb{R}^3$ defined by $x^2 + y^2 = 2 + z - e^{2z}$ for $z \geq 0$, oriented by a right-hand rule, and let $\vec{\mathsf{F}}(x, y, z) = \langle x^2 y, -xy^2, xyz \rangle$. Evaluate

$$\iint_{\mathcal{S}} \nabla \times \vec{\mathsf{F}} \cdot d\vec{\mathsf{S}}.$$



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Solution: Parametrizing $\mathcal{S}$ and doing the integral directly is very hard.

However, since $\partial\mathcal{S}$ is just the unit circle $\mathcal{C}$ ($x^2 + y^2 = 1$, $z = 0$), we can use Stokes' Theorem to replace $\mathcal{S}$ with the unit disk $\mathcal{D}$ (oriented upwards), because $\partial\mathcal{S} = \partial\mathcal{D}$.

Example 3 continued: Since $\mathcal{D}$ is a region in the plane, the surface integral over $\mathcal{D}$ is just a double integral, which we can calculate using polar coordinates:

$$\underbrace{\iint_{\mathcal{S}} \nabla \times \vec{F} \cdot d\vec{S}}_{= \oint_{\partial \mathcal{S}} \vec{F} \cdot d\vec{r}} = \underbrace{\iint_{\mathcal{D}} \nabla \times \vec{F} \cdot d\vec{S}}_{= \oint_{\partial \mathcal{D}} \vec{F} \cdot d\vec{r}}$$

$$= \iint_{\mathcal{D}} \left\langle xz,\ -yz,\ -x^2 - y^2 \right\rangle \cdot \vec{k}\, dA$$

$$= -\iint_{\mathcal{D}} x^2 + y^2\, dA$$

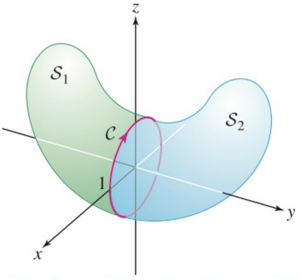
$$= -\int_0^{2\pi} \int_0^1 r^3\, dr\, d\theta$$

$$= -\pi/2.$$

Example 4: Let $\vec{F} = \text{curl}(\vec{A})$, where

$$\vec{A}(x, y, z) = \langle y + z, \sin(xy), e^{xyz} \rangle .$$

Find the flux of $\vec{F}$ outward through each of the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ whose common boundary $\mathcal{C}$ is the unit circle in the xz-plane.



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Solution: Parametrize $\mathcal{C}$ as $\vec{r}(t) = \langle \cos(t), 0, \sin(t) \rangle$.

The orientation of $\mathcal{C}$ has the surface $\mathcal{S}_1$ on the left.

$$\iint_{\mathcal{S}_1} \vec{F} \cdot d\vec{S} = \oint_{\mathcal{C}} \vec{A} \cdot d\vec{r} = \int_0^{2\pi} -\sin^2(t) + \cos(t)\, dt = -\pi.$$

The orientation of $\mathcal{C}$ has the surface $\mathcal{S}_2$ on the right.

$$\iint_{\mathcal{S}_2} \vec{F} \cdot d\vec{S} = \oint_{-\mathcal{C}} \vec{A} \cdot d\vec{r} = -\oint_{\mathcal{C}} \vec{A} \cdot d\vec{r} = \pi.$$

Note that we did not need to know what $\mathcal{S}_1$ and $\mathcal{S}_2$ were!